**Question 1**:

(a)(i) Find the derivative of $x^2 \cos x$. 2

(ii) Evaluate $\int_1^6 \frac{x}{x^2+4}\,dx$. 2

(b)(i) Sketch $y = |x+1|$. 2

(ii) Hence or otherwise solve $|x+1| = 3x$. 1

(c) If $f(x) = 2\sin^{-1}(3x)$, find

(i) the domain and range of $f(x)$, 2

(ii) $f\left(\frac{1}{6}\right)$, 1

(iii) $f'\left(\frac{1}{6}\right)$. 2

**QUESTION 2**: **(START A NEW PAGE)**

(a) P(-7,3), Q(9,15) and B(14,0) are three points and A divides the interval PQ in the ratio 3:1. Prove that PQ is perpendicular to AB. 3

(b) By using the substitution $u^2 = x+1$ evaluate $\int_0^3 \frac{x+2}{\sqrt{x+1}}\,dx$. 3

(c) Water flows from a hole in the base of a cylindrical vessel at a rate given by 6

$$\frac{dh}{dt} = -k\sqrt{h}$$

where $k$ is a constant and $h$ mm is the depth of water at time $t$ minutes.
If the depth of water falls from 2500mm to 900mm in 5 minutes, find how much longer it will take to empty the vessel.

## QUESTION 3: (START A NEW PAGE)

(a) Find the value of the constant term in the expansion of $\left(3x+\frac{2}{\sqrt{x}}\right)^6$. 3

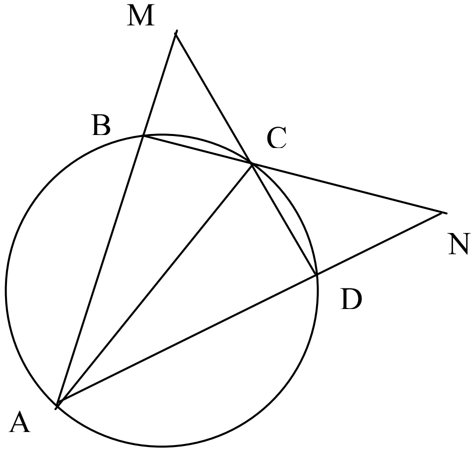
(b) Three boys (Adam, Bruce, Chris) and three girls (Debra, Emma, Fay) form a single queue at random in front of the school canteen window. Find the probability that:

(i) the first two to be served are Emma and Adam in that order, 2

(ii) a boy is at each end of the queue, 1

(iii) no two girls stand next to each other. 1

(c) In the figure *ABM*, *DCM*, *BCN* and *ADN* are straight lines and $\angle AMD = \angle BNA$.



(i) Copy the diagram onto your answer sheet and prove that $\angle ABC = \angle ADC$. 3

(ii) Hence prove that AC is a diameter. 2

## QUESTION 4: (START A NEW PAGE)

(a)(i) Given that $\sin^2 A+\cos^2 A=1$, prove that $\tan^2 A=\sec^2 A-1$. 2

(ii) Sketch the curve $y=4\tan^{-1}x$ clearly showing its range. 2

(iii) Find the volume of the solid formed when the area bounded by the curve $y=4\tan^{-1}x$, the *y*-axis and the line $y=\pi$ is rotated one revolution about the *y*-axis. 2

(b)(i) An object has velocity $v\ ms^{-1}$ and acceleration $\ddot{x}\ ms^{-2}$ at position $x\ m$ from the origin, show that $\frac{d}{dx}\left(\frac{1}{2}v^2\right)=\ddot{x}$. 2

(ii) The acceleration (in $ms^{-2}$) of an object is given by $\ddot{x}=2x^3+4x$.

(α) If the object is initially 2 *m* to the right of the origin traveling with velocity 6 $ms^{-1}$, find an expression for $v^2$ (the square of its velocity) in terms of *x*. 2

(β) What is the minimum speed of the object? (Give a reason for your answer) 2

## QUESTION 5: (START A NEW PAGE)

(a) The curves $y = e^{-2x}$ and $y = 3x + 1$ meet on the y-axis. Find the size of the acute angle between these curves at the point where they meet. 3

(b)(i) Sketch the function $y = f(x)$ where $f(x) = (x-1)^2 - 4$ clearly showing all intercepts with the co-ordinate axes. (Use the same scale on both axes) 2

(ii) What is the largest positive domain of $f$ for which *f(x)* has an inverse $f^{-1}(x)$? 1

(iii) Sketch the graph of $y = f^{-1}(x)$ on the same axes as (i). 1

(c) In tennis a player is allowed a maximum of two serves when attempting to win a point. If the first serve is not legal it is called a fault and the server is allowed a second serve. If the second serve is also illegal then it is called a double fault and the server loses the point. The probability that Pat Smash's first serve will be legal is 0.4. If Pat Smash needs to make a second serve then the probability that it will be legal is 0.7.

(i) Find the probability that Pat Smash will serve a double fault when trying to win a point. 2

(ii) If Pat Smash attempts to win six points, what is the probability that he will serve at least two double faults? (Give answer correct to 2 decimal places) 3

## QUESTION 6: (START A NEW PAGE)

(a) A spherical bubble is expanding so that its volume is increasing at $10\ cm^3 s^{-1}$. 3

Find the rate of increase of its radius when the surface area is $500\ cm^2$.

(Volume = $\frac{4}{3}\pi r^3$ , Surface area = $4\pi r^2$ )

(b) Prove by Mathematical Induction that: 4

$$2(1!) + 5(2!) + 10(3!) + \ldots + (n^2 + 1)n! = n(n+1)! \text{ for positive integers } n \geq 1.$$

(c) If $y = \dfrac{\log_e x}{x}$ find $\dfrac{dy}{dx}$ and hence show that $\displaystyle\int_e^{e^2} \frac{1 - \log_e x}{x \log_e x}\, dx = \log_e 2 - 1$. 5

**QUESTION 7**: **(START A NEW PAGE)**

(i) By considering the expansion of $\sin(X+Y)-\sin(X-Y)$ prove that 3

$$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right).$$

(ii) Also given that $\cos A - \cos B = 2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{B-A}{2}\right)$ prove that 2

$$\frac{\sin A - \sin B}{\cos A - \cos B} = -\cot\left(\frac{A+B}{2}\right).$$

(iii) Prove that the position of a projectile t seconds after projection from ground level with initial horizontal and vertical velocity components of $V\cos\alpha$ and $V\sin\alpha$ respectively is given by $x = Vt\cos\alpha$ and $y = -\frac{1}{2}gt^2 + Vt\sin\alpha$. (Assume that there is no air resistance) 2

(iv) Two objects P and Q are projected from the same ground position at the same time with initial speed $V\ ms^{-1}$ at angles $\alpha$ and $\beta$ respectively ($\beta > \alpha$).

($\alpha$) If at time t seconds the line joining P and Q makes an acute angle $\theta$ with the horizontal prove that $\tan\theta = \left|\frac{\sin\beta - \sin\alpha}{\cos\beta - \cos\alpha}\right|$. 3

($\beta$) Hence show that $\theta = \frac{1}{2}(\pi - \alpha - \beta)$. 2

**THIS IS THE END OF THE EXAMINATION PAPER**

# JRAHS 3UNIT TRIAL 2001

## QUESTION 1

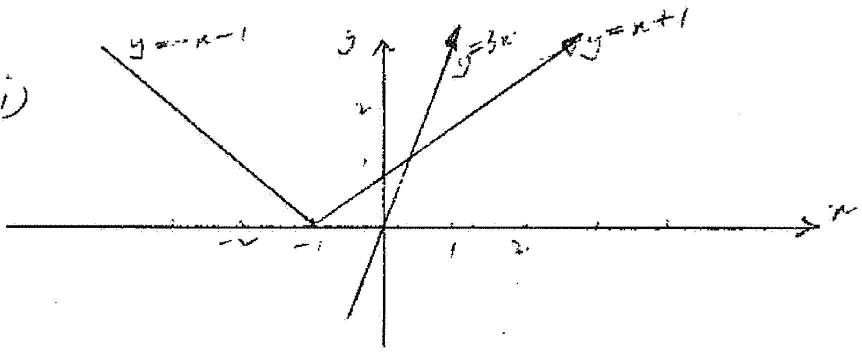
(a)(i) $y' = 2x\cos x - x^2 \sin x$

(ii) $\int_1^6 \frac{x}{x^2+4}\,dx = \frac{1}{2}\left[\ln(x^2+4)\right]_1^6$

$= \frac{1}{2}(\ln 40 - \ln 5)$

$= \frac{1}{2}\ln 8$

(b)(i)



(ii) $3x = x+1$ (from graph)

$2x = 1$

$x = \frac{1}{2}$

(c)(i) Domain $-\frac{1}{3} \le x \le \frac{1}{3}$

Range $-\pi \le y \le \pi$

(ii) $f(\frac{1}{6}) = 2\sin^{-1}(\frac{3}{6})$

$= \pi/3$

(iii) $f'(x) = 2 \cdot \frac{3}{\sqrt{1-9x^2}}$

$f'(\frac{1}{6}) = \frac{6}{\sqrt{1-9/36}}$

$= \frac{6}{\sqrt{3/4}}$

$= \frac{12}{\sqrt{3}}$ or $4\sqrt{3}$

## QUESTION 2.

(a) $P(-7,3)$ $Q(9,15)$ $B(14,0)$

$\times$

$3:1$

$$A\left(\frac{-7+27}{4}, \frac{3+45}{4}\right) = A(5,12)$$

$$m(PQ) = \frac{15-3}{9+7}$$

$$= 3/4$$

$$m(AB) = \frac{12-0}{5-14}$$

$$= -4/3$$

$$m(PQ)\cdot m(AB) = 3/4 \cdot -4/3$$

$$= -1$$

$\therefore PQ \perp AB$ (prod. of slopes $= -1$)

(b) $x=0$ $u^2=1$ $x=3$, $u^2=4$

$u=1$ (take $u>0$) $u-2$ (take $u>0$)

$$u^2-1=x$$

$$\frac{dx}{du} = 2u$$

$$dx = 2u\,du$$

$$\therefore \int_0^3 \frac{x+2}{\sqrt{x+1}}\,dx = \int_1^2 \frac{u^2+1}{\sqrt{u^2}}\,2u\,du$$

$$= 2\int_1^2 \frac{u^2+1}{u}\,du$$

$$= 2\int_1^2 u+\frac{1}{u}\,du$$

$$= 2\left[\frac{1}{2}u^2+\ln u\right]_1^2$$

$$= 2\left\{\left(4/2+\ln 2\right)-\left(1/2+\ln 1\right)\right\}$$

$$= 3+2\ln 2$$

$6\frac{2}{3}$

2(c) $\frac{dt}{dh} = -\frac{1}{k} \cdot h^{-\frac{1}{2}}$

$t = -\frac{1}{k} \cdot 2h^{\frac{1}{2}} + c$

$t = \frac{-2\sqrt{h}}{k} + c$

$t = 0 \quad h = 2500$

$0 = \frac{-100}{k} + c$ — ①

$t = 5 \quad h = 900$

$5 = -\frac{60}{k} + c$ — ②

② − ① $\quad 5 = \frac{-60}{k} + \frac{100}{k}$

$5k = 40$

$k = 8$

from ① $\quad c = \frac{100}{8}$

$\therefore t = \frac{-\sqrt{h}}{4} + 12.5$

when $h = 0 \quad t = 12.5$

$\therefore$ extra time $= 12.5 - 5$

$= 7.5$ min.

QUESTION 3.

(a) $T_{r+1} = {}^6C_r \, (3x)^{6-r} \left(\frac{2}{\sqrt{x}}\right)^r$

$= {}^6C_r \, 3^{6-r} \, 2^r . \, x^{6-r} \, x^{-\frac{1}{2}r}$

$= {}^6C_r \, 3^{6-r} \, 2^r \, x^{6-\frac{1}{2}r}$

for constant term degree of $x = 0$

$\therefore \quad 6 - \frac{1}{2}r = 0$

$\frac{3}{2} r = 6$

$r = 4$

$\therefore$ constant $= {}^6C_4\, 3^2 . 2^4$

$= 2160$

(b) (i) ① ① ④ ③ ② ①

E A

Prob $= \frac{4!}{6!}$ $\underline{\underline{\text{or}}}$ Prob $= \frac{1}{6} \cdot \frac{1}{5}$

$= \frac{1}{30}$ $= \frac{1}{30}$

(ii) C E

③ ④ ③ ② ① ②

Prob $= \frac{3.2.4!}{6!}$

$= \frac{1}{5}$

(iii) 2 ③ 4 ② 3 ①

G B G B G B

(Place Boys then fill gaps with girls)

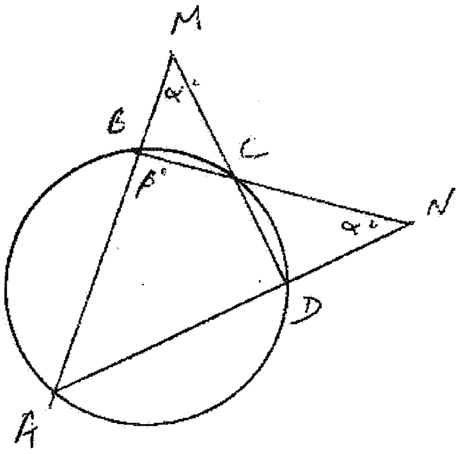
Prob $= \frac{3!\, 4.3.2}{6!}$

$= \frac{1}{5}$

(C) (i) let $A\hat{M}D = A\hat{N}B = \alpha°$

& $A\hat{B}C = \beta°$

$B\hat{C}M = (\beta - \alpha)°$ (exterior angle of $\triangle BMC$ equals sum of opposite interior angles)



$D\hat{C}N = (\beta - \alpha)°$ (vertically opposite angles)

$A\hat{D}C = \beta°$ (exterior angle of $\triangle CND$ equals sum of opposite interior angles

$\therefore A\hat{B}C = A\hat{D}C$ ([illegible]

Q3(c)(ii) $\hat{ABC} + \hat{ADC} = 180°$ (opposite angles of cyclic quad ABCD are supplementary)

$2\hat{ABC} = 180°$ ( $\hat{ABC} = \hat{ADC}$ , part (i))

$\hat{ABC} = 90°$

$\therefore$ AC is a diameter ( angle in semicircle is 90° )

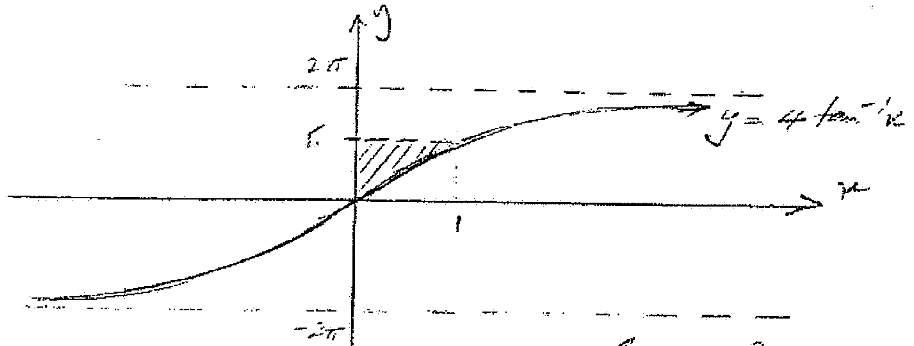
QUESTION 4

(a)(i) $\frac{\sin^2 A}{\cos^2 A} + \frac{\cos^2 A}{\cos^2 A} = \frac{1}{\cos^2 A}$

$\tan^2 A + 1 = \sec^2 A$

$\tan^2 A = \sec^2 A - 1$

(ii)



Range $-2\pi < y < 2\pi$

(iii) $y/4 = \tan^{-1}x$

$x = \tan y/4$

$$V = \pi\int_0^{\pi} x^2\,dy$$

$$= \pi\int_0^{\pi} \tan^2 y/4\,dy$$

$$= \pi\int_0^{\pi} \sec^2 y/4 - 1\,dy$$

$$= \pi\left[4\tan y/4 - y\right]_0^{\pi}$$

$$= \pi\left\{(4\tan \pi/4 - \pi) - (4\tan 0 - 0)\right\}$$

$$= \pi(4-\pi)\ u^3.$$

(b)(i) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dv}\left(\frac{1}{2}v^2\right) \cdot \frac{dv}{dx}$

$= v\frac{dv}{dx}$

$= \frac{dx}{dt} \cdot \frac{dv}{dx}$

$= \frac{dv}{dt}$

$= \ddot{x}$

(ii)(α) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 2x^3 + 4x$

$\frac{1}{2}v^2 = \frac{x^4}{2} + 2x^2 + c$

$t = 0,\ x = 2,\ v = 6$

$18 = 8 + 8 + c$

$c = 2$

$v^2 = \_\,x^4 + 4x^2 + 4$

(β) $v^2 = (x^2+2)^2 \quad \therefore v^2 \geq 4 \quad \therefore v \neq 0$

∴ object never changes direction

∴ always moves to right with increasing speed

since initial vel > 0 & accel > 0 for x > 0

∴ min speed is initial speed

∴ min. speed = 6 $ms^{-1}$

## QUESTION 5

(a) $y' = -2e^{-2x}$

when $x = 0$, $y' = -2e^0$

$\therefore m_1 = -2$

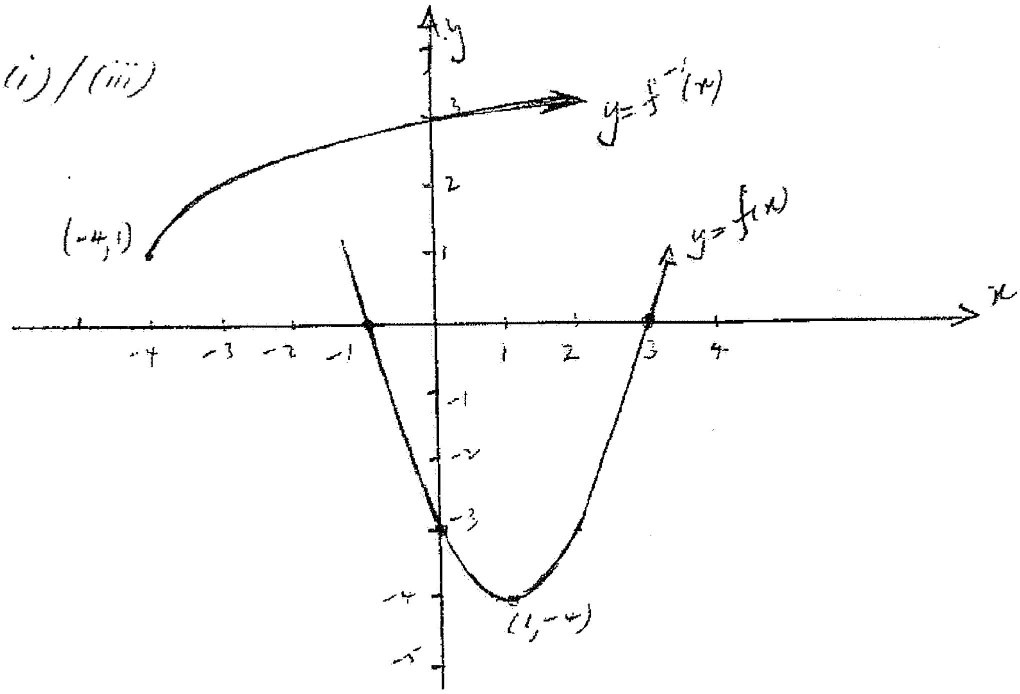
$m_2 = 3$

$\tan\theta = \left|\frac{3+2}{1+(3)(-2)}\right|$

$= 1$

$\theta = \pi/4$, or $45°$

(b) (i)/(iii)

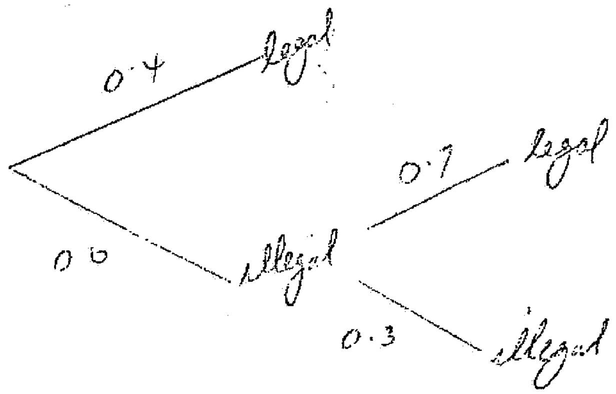


(ii) $x \geq 1$

(iii) see graph.

(c)

(i)



$$P(\text{double fault}) = 0\cdot6 \times 0\cdot3$$
$$= 0\cdot18$$

(ii) $(0\cdot82 + 0\cdot18)^6$

P(at least 2 double faults) = 1 − { P(0 double faults) + P(1 double fault

$$= 1 - \left\{ {}^6C_0 (0\cdot82)^6 (0\cdot18)^0 + {}^6C_1 (0\cdot82)^5 (0\cdot18)^1 \right\}$$

$$\doteqdot 0\cdot30.$$

# QUESTION 6.

(a) $\frac{dr}{dt} = \frac{dr}{dV} \cdot \frac{dV}{dt}$

$= \frac{1}{4\pi r^2} \cdot 10$

$= \frac{10}{4\pi r^2}$

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$

when $SA = 500$ $(= 4\pi r^2)$

$\frac{dr}{dt} = \frac{10}{500}$

$= \frac{1}{50}$ cm/s.

(b) when $n=1$, $LHS = 2(1!)$ $= 2$ $RHS = 1(2!)$ $= 2$.

$\therefore$ true for $n=1$

assume true for $n=k$

ie. $2(1!) + 5(2!) + \cdots + (k^2+1)k! = k(k+1)!$

to prove true for $n=k+1$

ie $2(1!) + 5(2!) + \cdots + (k^2+1)k! + [(k+1)^2+1](k+1)! = (k+1)(k+2)!$

Now $LHS = 2(1!) + 5(2!) + \cdots + (k^2+1)k! + (k^2+2k+2)(k+1)!$

$= k(k+1)! + (k^2+2k+2)(k+1)!$ $\underline{\underline{\text{(by assumption)}}}$ Q??

$= (k+1)!\{k + k^2 + 2k + 2\}$

$= (k+1)!(k^2+3k+2)$

$= (k+1)!(k+2)(k+1)$

$= (k+2)!(k+1)$

$= RHS$

$\therefore$ if true for $n=k$ then true for $n=k+1$ &
since true for $n=1$ then true for all
$n \geq 1$.

Q6(c) $$\frac{dy}{dx} = \frac{(x)(1/x) - (1)(\ln x)}{x^2}$$

$$= \frac{1 - \ln x}{x^2}$$

$$\int_e^{e^2} \frac{1 - \ln x}{x \ln x}\, dx = \int_e^{e^2} \frac{\frac{1 - \ln x}{x^2}}{\frac{\ln x}{x}}\, dx$$

$$= \left[\ln\left(\frac{\ln x}{x}\right)\right]_e^{e^2}$$

$$= \ln\left(\frac{\ln e^2}{e^2}\right) - \ln\left(\frac{\ln e}{e}\right)$$

$$= \ln(2/e^2) - \ln(1/e)$$

$$= \ln\left(\frac{2}{e^2} \times \frac{e}{1}\right)$$

$$= \ln(2/e)$$

$$= \ln 2 - \ln e$$

$$= \ln 2 - 1$$

## QUESTION 7

(i) $\sin(x+y) - \sin(x-y) = (\sin x \cos y + \cos x \sin y) - (\sin x \cos y - \cos x \sin y)$ ①

$= 2 \cos x \sin y$

let $x + y = A$ & $x - y = B$

$\therefore 2x = A + B$ $\qquad 2y = A - B$

$x = \frac{A+B}{2}$ ① $\qquad y = \frac{A-B}{2}$ ①

$$\therefore \sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

(ii) $$\frac{\sin A - \sin B}{\cos A - \cos B} = \frac{2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}{2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{B-A}{2}\right)}$$ ①

$$= \frac{\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}{-\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}$$ ①

$$= -\cot\left(\frac{A+B}{2}\right)$$

(iii) $\ddot{x}=0$

$\dot{x}=c_1$

$t=0,\ \dot{x}=V\cos\alpha$

$\therefore V\cos\alpha=c_1$

$\dot{x}=V\cos\alpha$

$x=Vt\cos\alpha+c_2$

$t=0,\ x=0\ \therefore c_2=0$

$x=Vt\cos\alpha$ ①

$\ddot{y}=-g$

$\dot{y}=-gt+c_3$

$t=0,\ \dot{y}=V\sin\alpha$

$V\sin\alpha=c_3$

$\dot{y}=-gt+V\sin\alpha$

$y=-\frac{1}{2}gt^2+Vt\sin\alpha+c_4$

$t=0,\ y=0\ \therefore c_4=0$

$y=-\frac{1}{2}gt^2+Vt\sin\alpha$ ①

(iv)($\alpha$) Particle P

$x_P=Vt\cos\alpha \qquad y_P=-\frac{1}{2}gt^2+Vt\sin\alpha$

Particle Q

$x_Q=Vt\cos\beta \qquad y_Q=-\frac{1}{2}gt^2+Vt\sin\beta$

$\tan\theta$ = slope PQ ①

$$=\left|\frac{(-\frac{1}{2}gt^2+Vt\sin\beta)-(-\frac{1}{2}gt^2+Vt\sin\alpha)}{Vt\cos\beta-Vt\cos\alpha}\right|$$

$$=\left|\frac{Vt(\sin\beta-\sin\alpha)}{Vt(\cos\beta-\cos\alpha)}\right| \quad ①$$

$$=\left|\frac{\sin\beta-\sin\alpha}{\cos\beta-\cos\alpha}\right| \quad ①$$

($\beta$) $\tan\theta=\left|-\cot\left(\frac{\alpha+\beta}{2}\right)\right|$ (cc) from (ii) (cc)

$$=\tan\left(\frac{\pi}{2}-\left(\frac{\alpha+\beta}{2}\right)\right)$$ ① $\alpha, \beta, \theta$ acute

$$\therefore \theta=\pi/2-\left(\frac{\alpha+\beta}{2}\right) \quad ①$$

$$\theta=\frac{1}{2}(\pi-\alpha-\beta)$$